Lecture 7
Pushdown Automata

COSE215: Theory of Computation

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Fall 2023

Contents

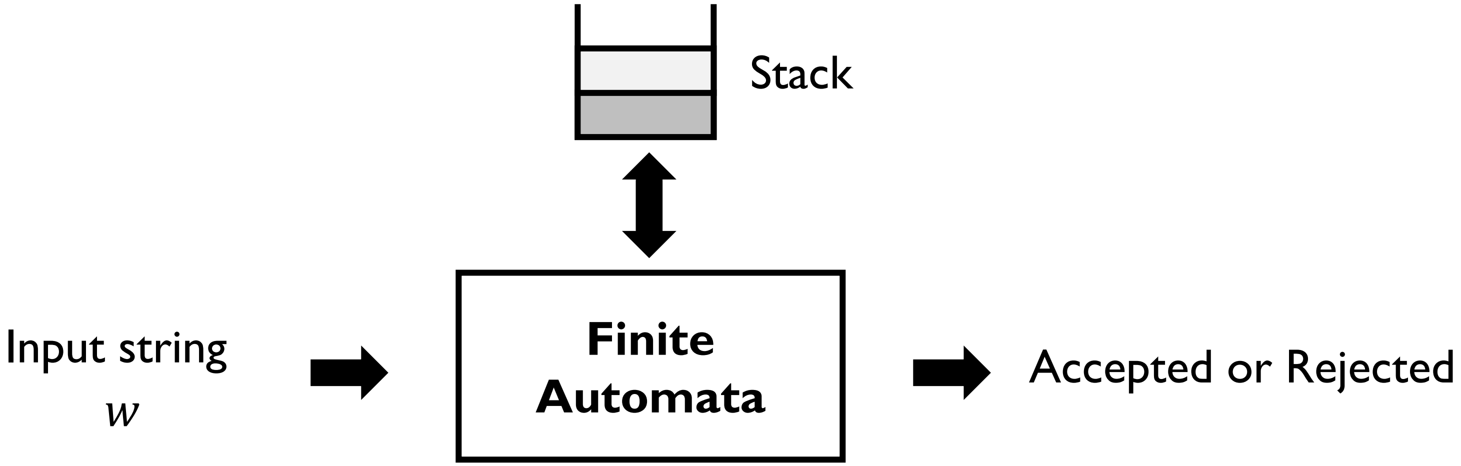
- **Pushdown Automata**

Limitation of finite automata

- **Finite automata cannot recognize context-free languages**
  - Because FA have limited memory and storage
  - E.g., $L = \{a^n b^n \mid n \geq 1\}$
    - FA cannot count the number of symbols contained in the input string

Pushdown Automata

- **Pushdown Automata (PDA)**
  - PDA are essentially finite automata
  - PDA have an extra component called a stack
    - ❖ A stack, by definition, has infinite length
    - ❖ This overcomes the limitation on finite automata arising from a bounded memory



Pushdown Automata

- **(Nondeterministic) Pushdown Automata: Formal definition**
  - A pushdown automaton (PDA) is a 7-tuple: $\boldsymbol{M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)}$
    - Q is a finite set of **internal states**
    - Σ is a finite set of **symbols**
    - Γ is a finite set of symbols called **stack alphabets**
    - δ is a set of **transition functions**
      - $\delta: Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow 2^{(Q \times \Gamma^*)}$
    - $q_0 \in Q$ is **the initial state**
    - $z \in \Gamma$ is the **initial stack alphabet**
    - $F \subseteq Q$ is a set of **final states**

Pushdown Automata

- **Example**
  - $M = (\{q_0, q_1, q_2\}, \{a, b\}, \{0, z\}, \delta, q_0, z, \{q_2\})$
    - $\delta(q_0, a, z) = \{(q_0, 0z)\}$
    - $\delta(q_0, a, 0) = \{(q_0, 00)\}$
    - $\delta(q_0, b, 0) = \{(q_1, \lambda)\}$
    - $\delta(q_1, b, 0) = \{(q_1, \lambda)\}$
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Pushdown Automata

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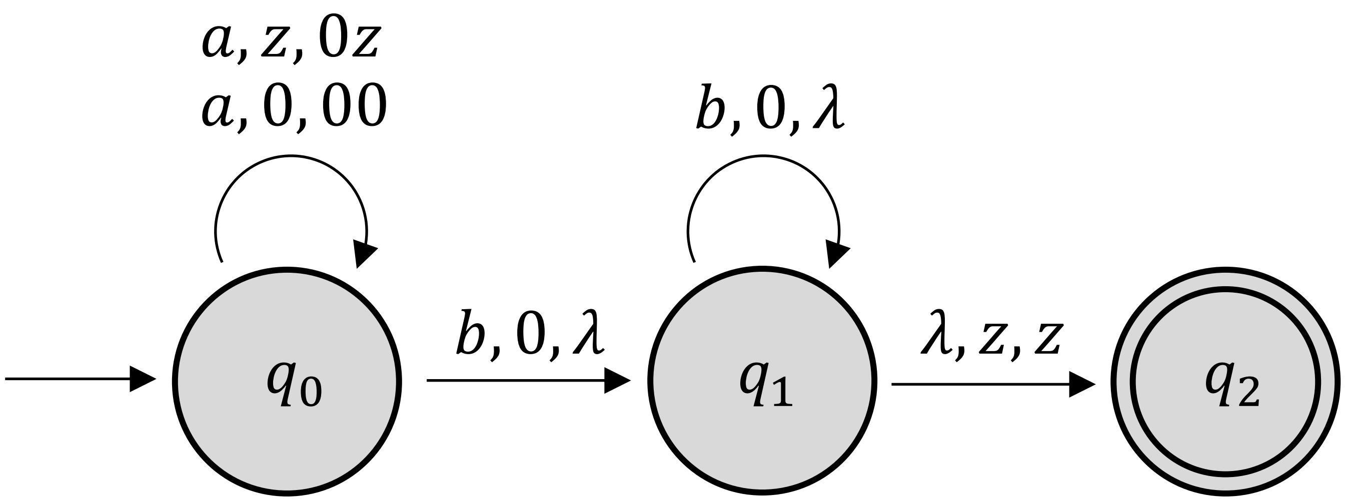

Pushdown Automata

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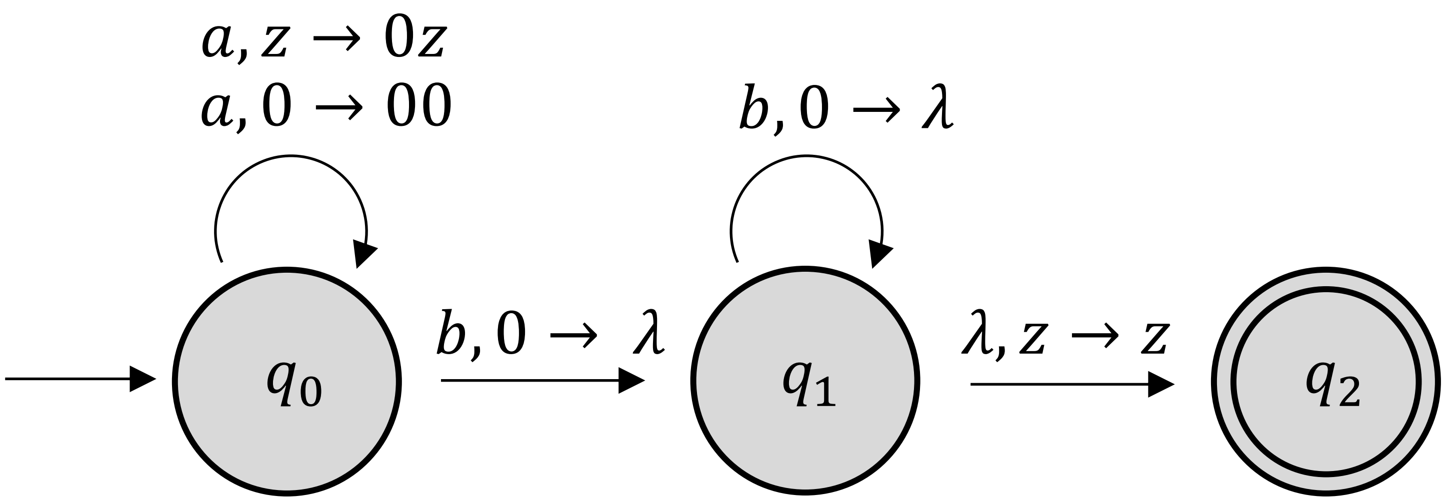

Pushdown Automata

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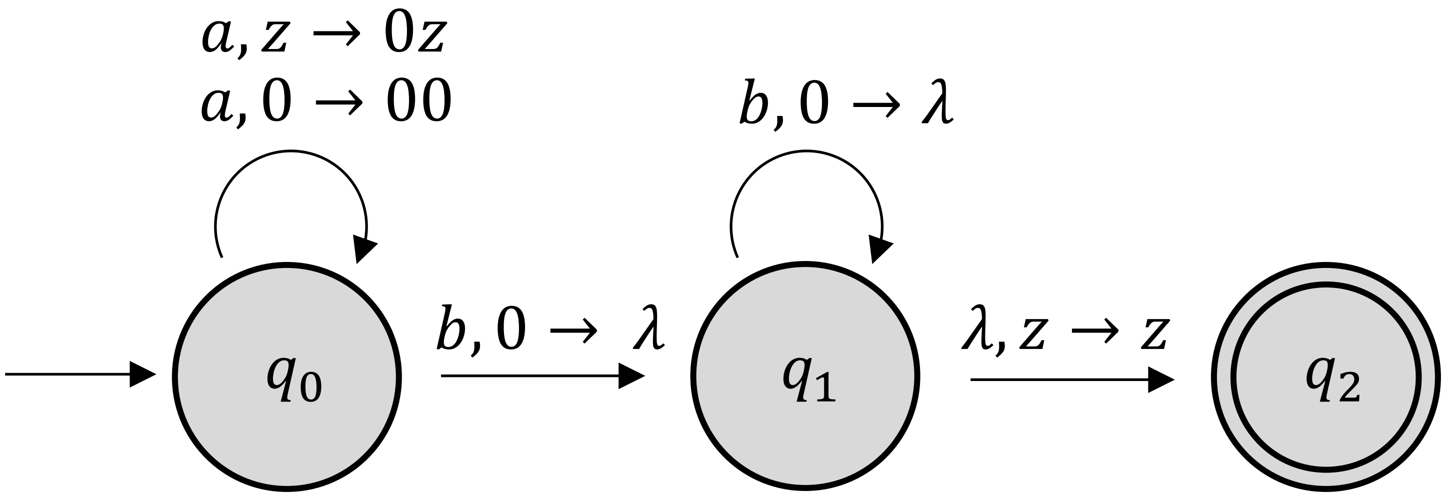
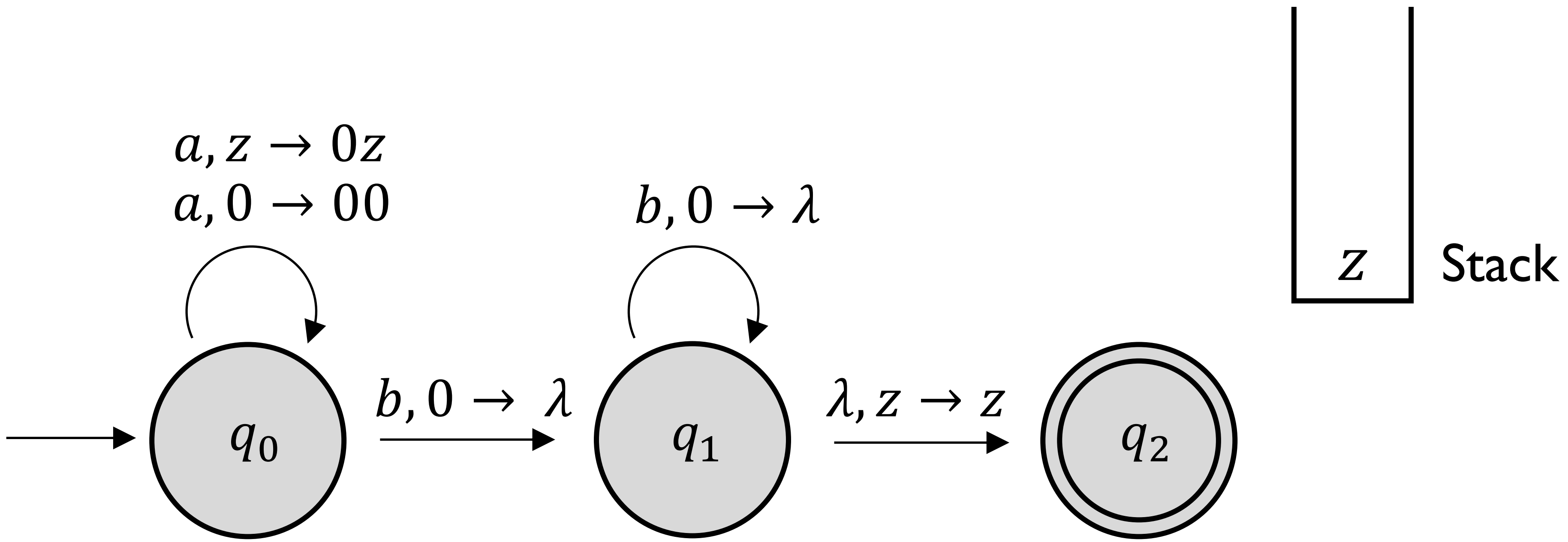
PDA for recognizing $L = \{a^n b^n \mid n \geq 1\}$



Pushdown Automata

- **Example**
  - E.g., Input string is "$aabb$"



Pushdown Automata

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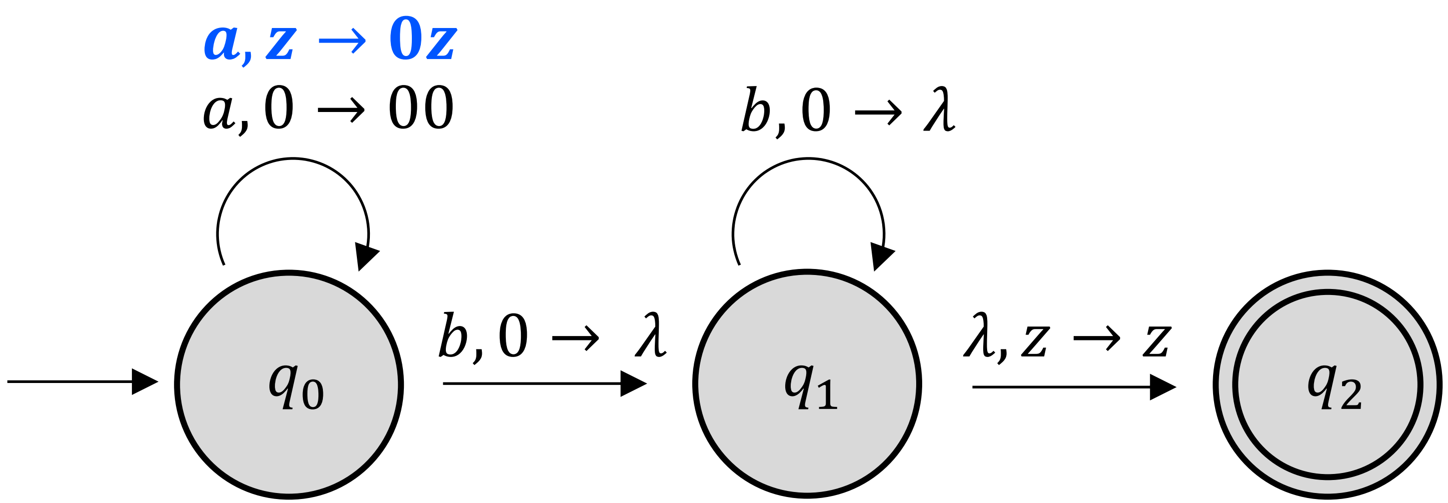
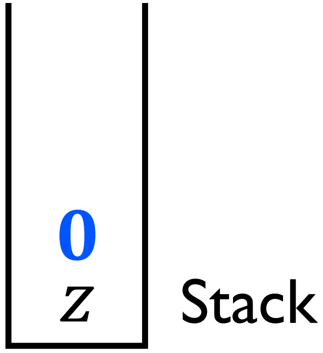
  - E.g., Input string is "$aabb$"

$$aabb$$

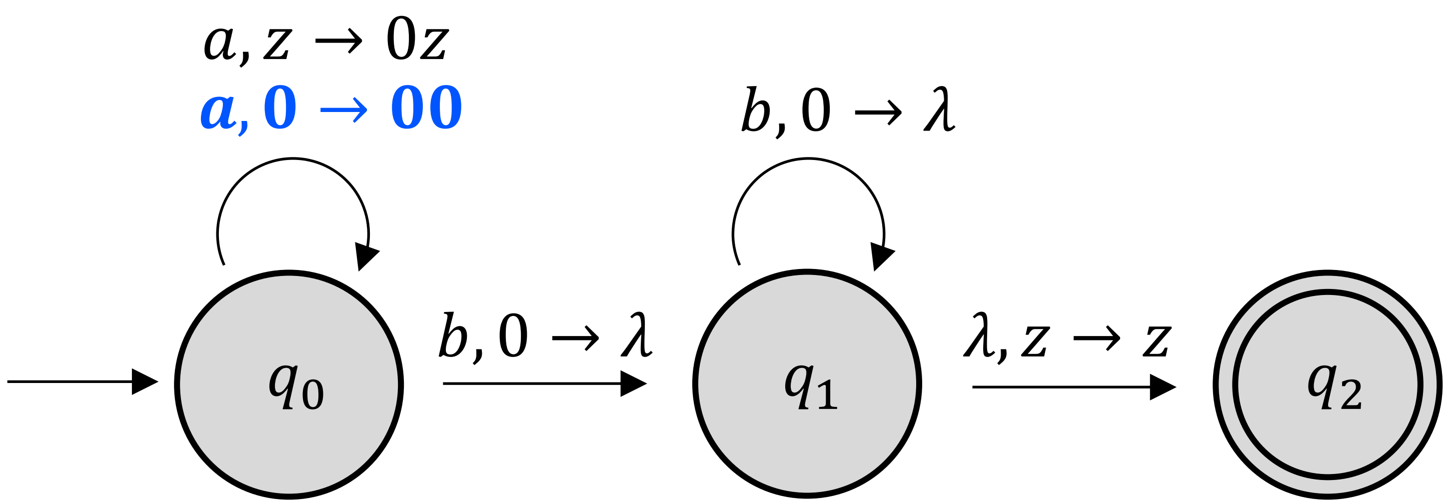


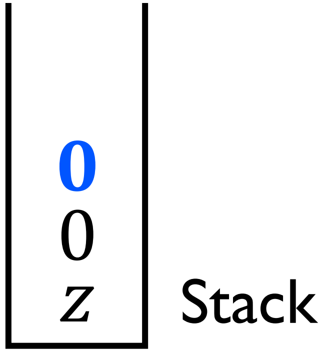


Pushdown Automata

- **Example**
  - E.g., Input string is "$aabb$"

$$\cancel{a}abb$$

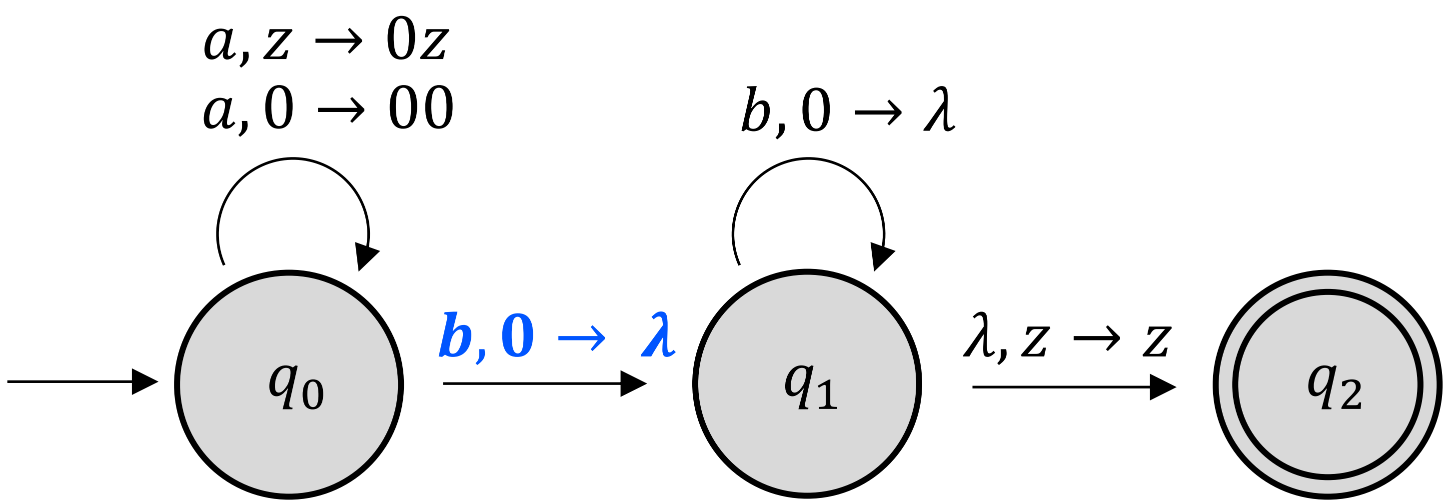


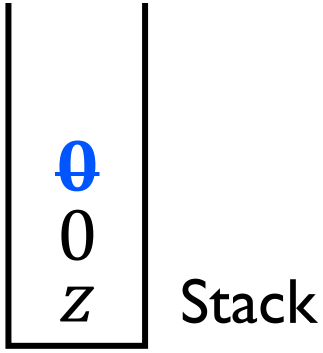


Pushdown Automata

- **Example**
  - E.g., Input string is "$aabb$"

$$\cancel{a}\cancel{a}bb$$

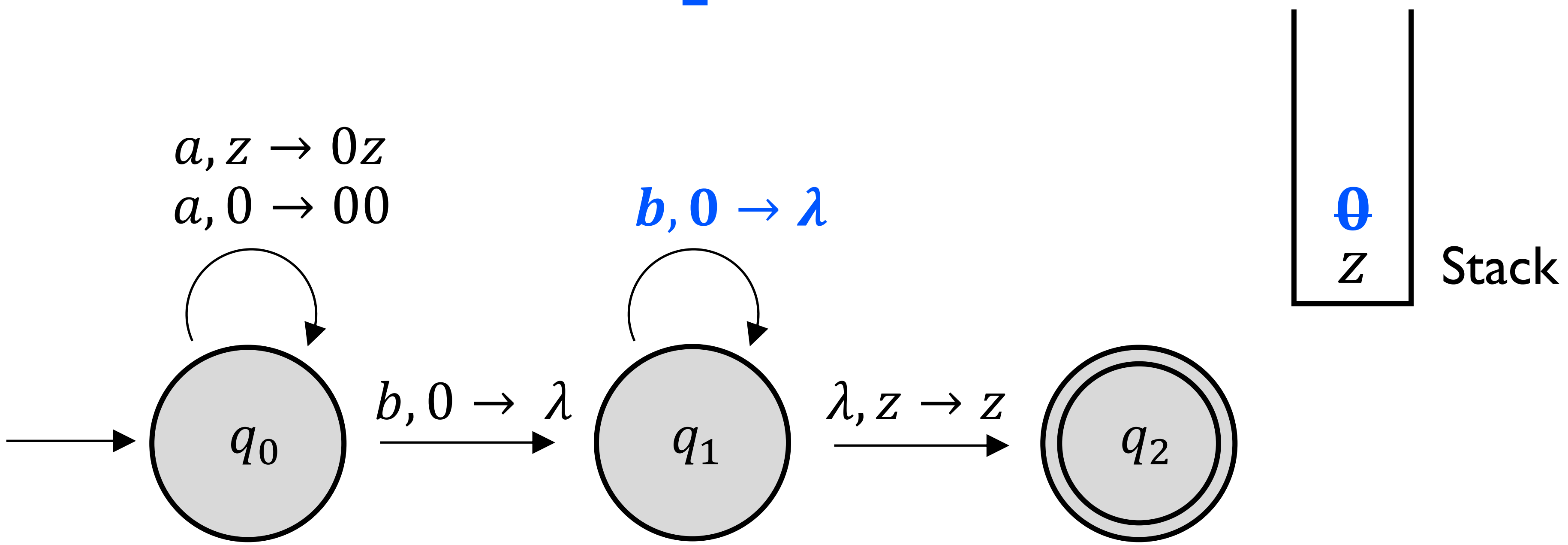




Pushdown Automata

- **Example**
  - E.g., Input string is "$aabb$"

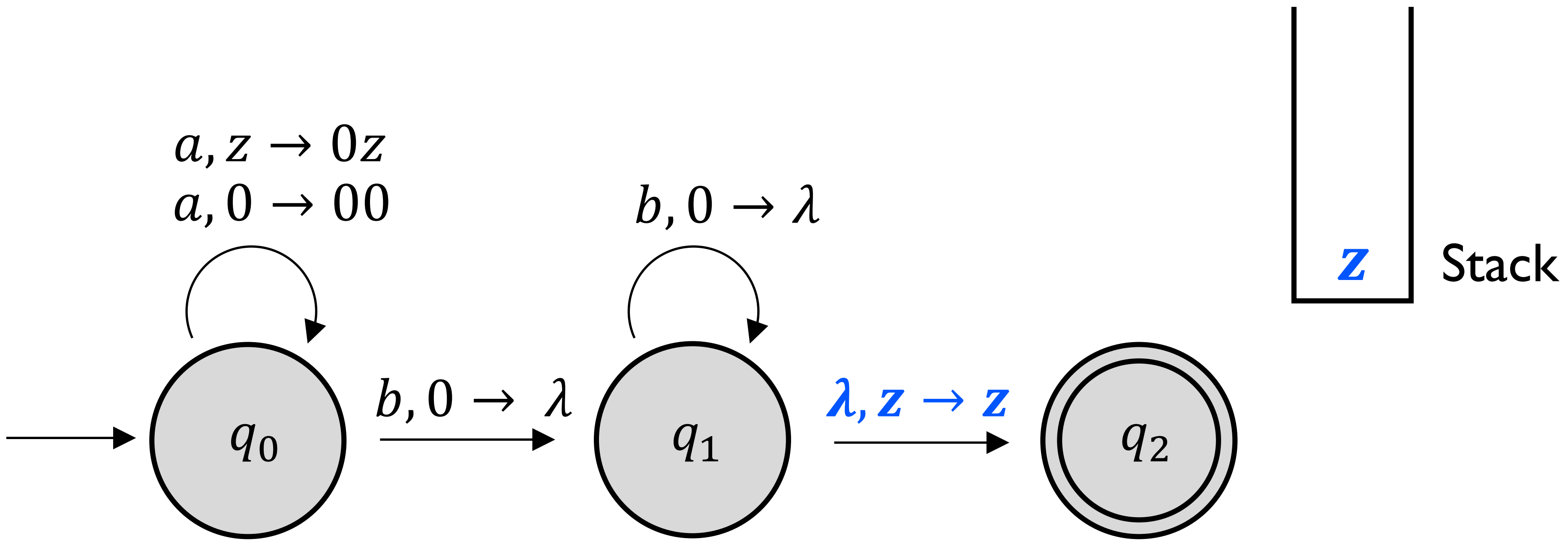
$\cancel{aab}b$



Pushdown Automata

- **Example**
  - E.g., Input string is "$aabb$"

$\cancel{aabb}$ ***Accepted!***

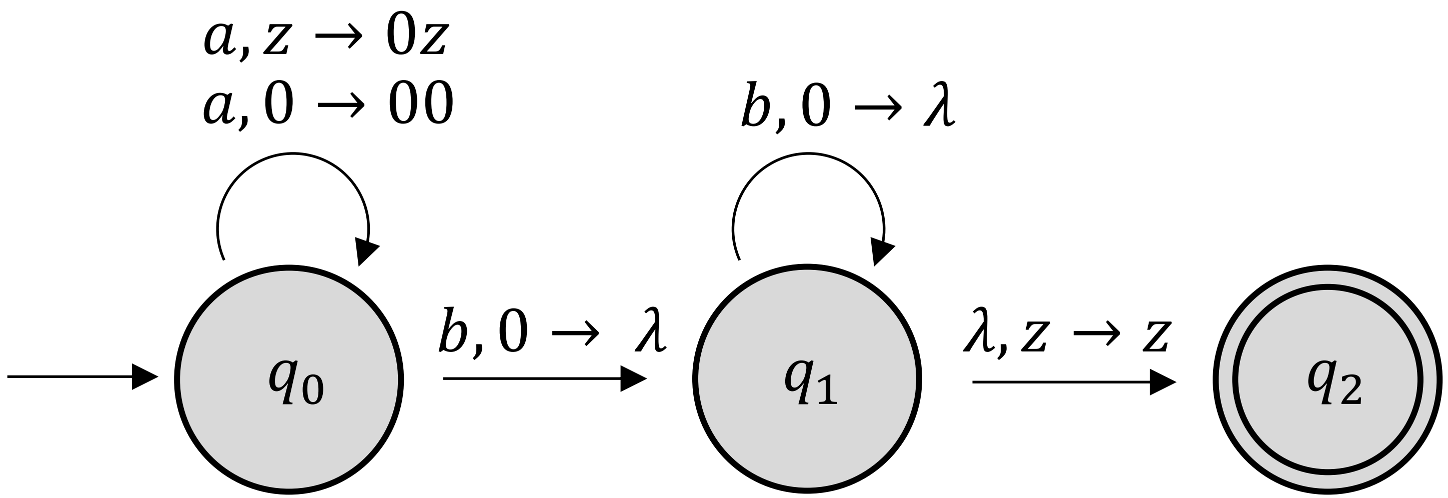


Pushdown Automata

- **Instantaneous description (ID)**
  - A notation for describing the configurations of PDA during the processing of a string
  - Represented as (q, w, u)
    - q: current state
    - w: unread part of the input string
    - u: stack contents (leftmost symbol represents the top of the stack)
  - A move from one ID to another is denoted by the symbol " $\vdash$ " (turnstile)
    - e.g., $(q_1, aw, bx) \vdash (q_2, w, yx)$

Pushdown Automata

- **Instantaneous description (ID): Example**
  - Processing "$aabb$"
  - $(q_0, aabb, z)$



Pushdown Automata

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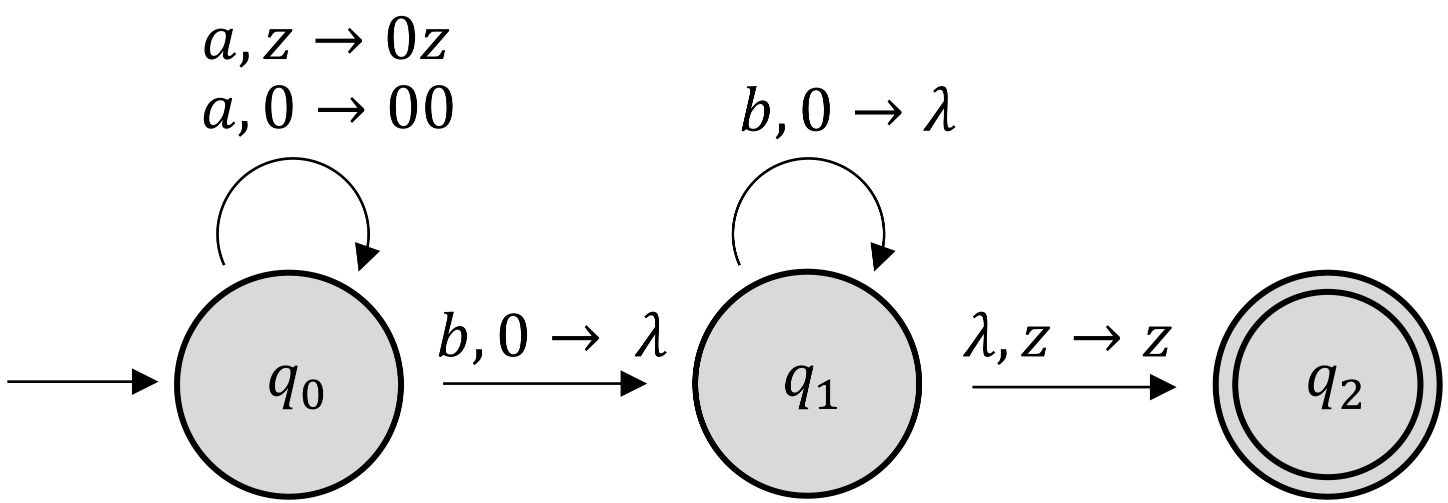

Pushdown Automata

- **Instantaneous description (ID): Example**
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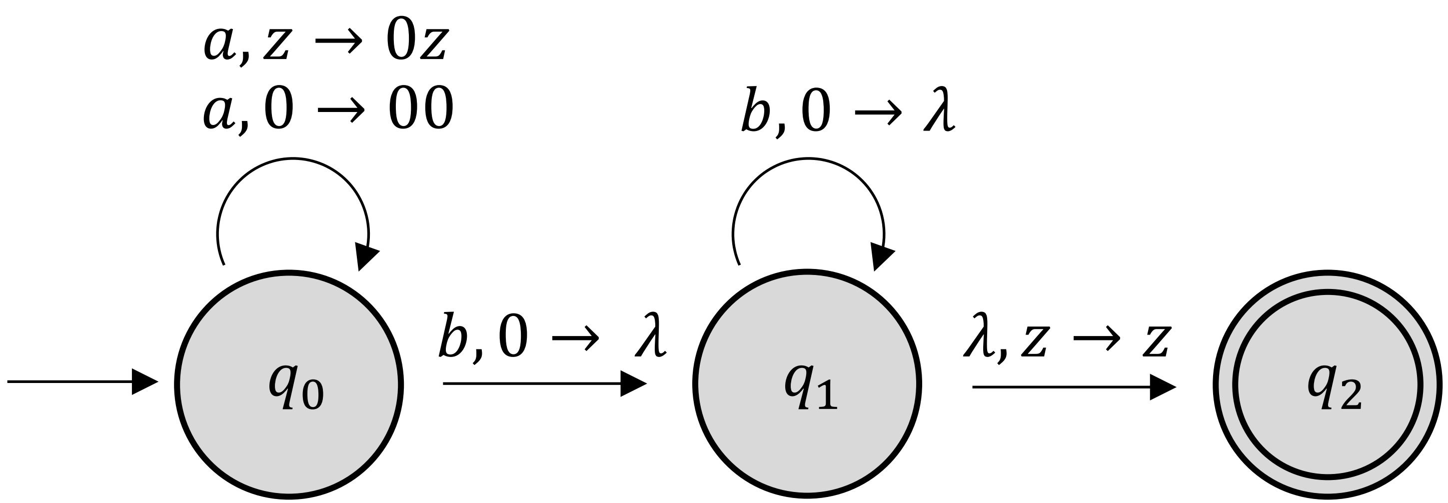
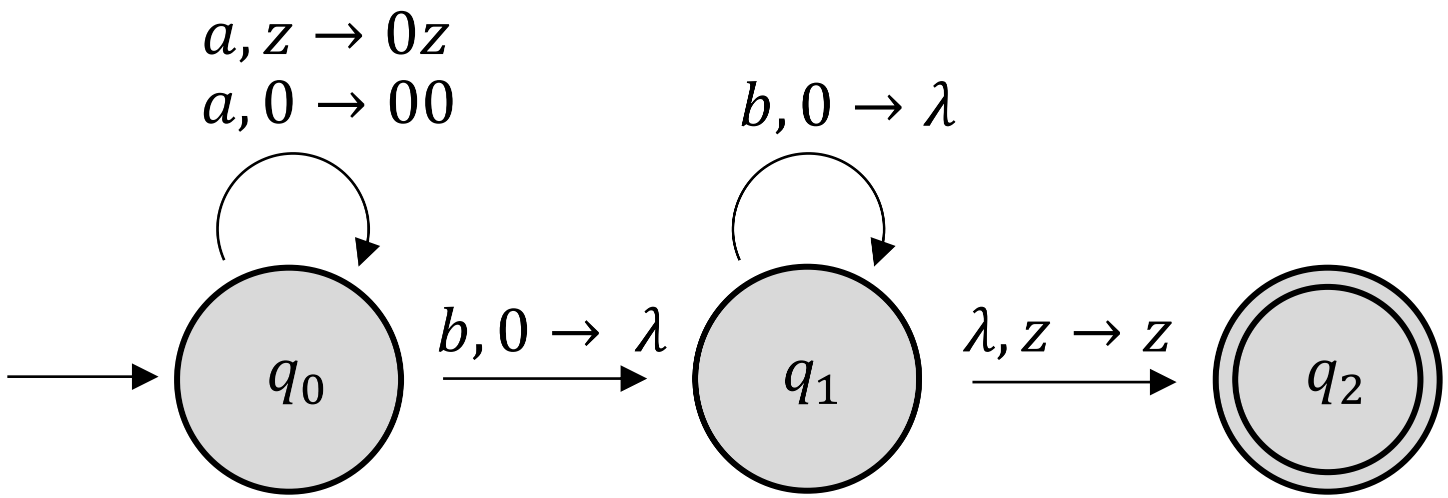
  - $(q_0, aabb, z) \vdash (q_0, abb, 0z) \vdash (q_0, bb, 00z)$



Pushdown Automata

- **Instantaneous description (ID): Example**
  - Processing "$aabb$"
  - $(q_0, aabb, z) \vdash (q_0, abb, 0z) \vdash (q_0, bb, 00z) \vdash (q_1, b, 0z) \vdash (q_1, \lambda, z) \vdash (q_2, \lambda, z)$



Pushdown Automata

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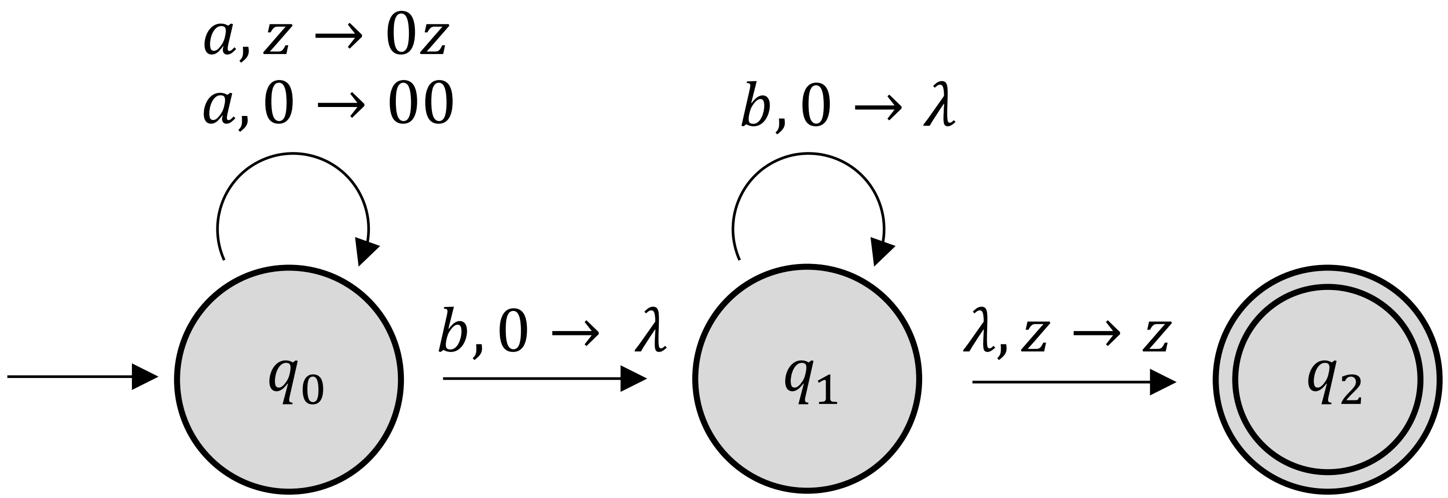
  - $(q_0, aabb, z) \vdash^* (q_2, \lambda, z)$



Pushdown Automata

- **The Language accepted by a PDA**
  - Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ be a nondeterministic pushdown automaton
  - The language accepted by M is the set

$$L(M) = \{w \in \Sigma^* : (q_0, w, z) \vdash^* (p, \lambda, u), p \in F, u \in \Gamma^*\}$$

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$$L(M) = \{w \in \Sigma^* : (q_0, w, z) \vdash^* (p, \lambda, u), p \in F, u \in \Gamma^*\}$$

  - Set of all strings that can put M into a final state at the end of the string
    - ❖ The final stack contents u is irrelevant to this definition of acceptance

Pushdown Automata

- **Another example:** Design a PDA for $L = \{ww^R : w \in \{a, b\}^*\}$

Pushdown Automata

- **Another example:** Design a PDA for $L = \{ww^R : w \in \{a, b\}^*\}$
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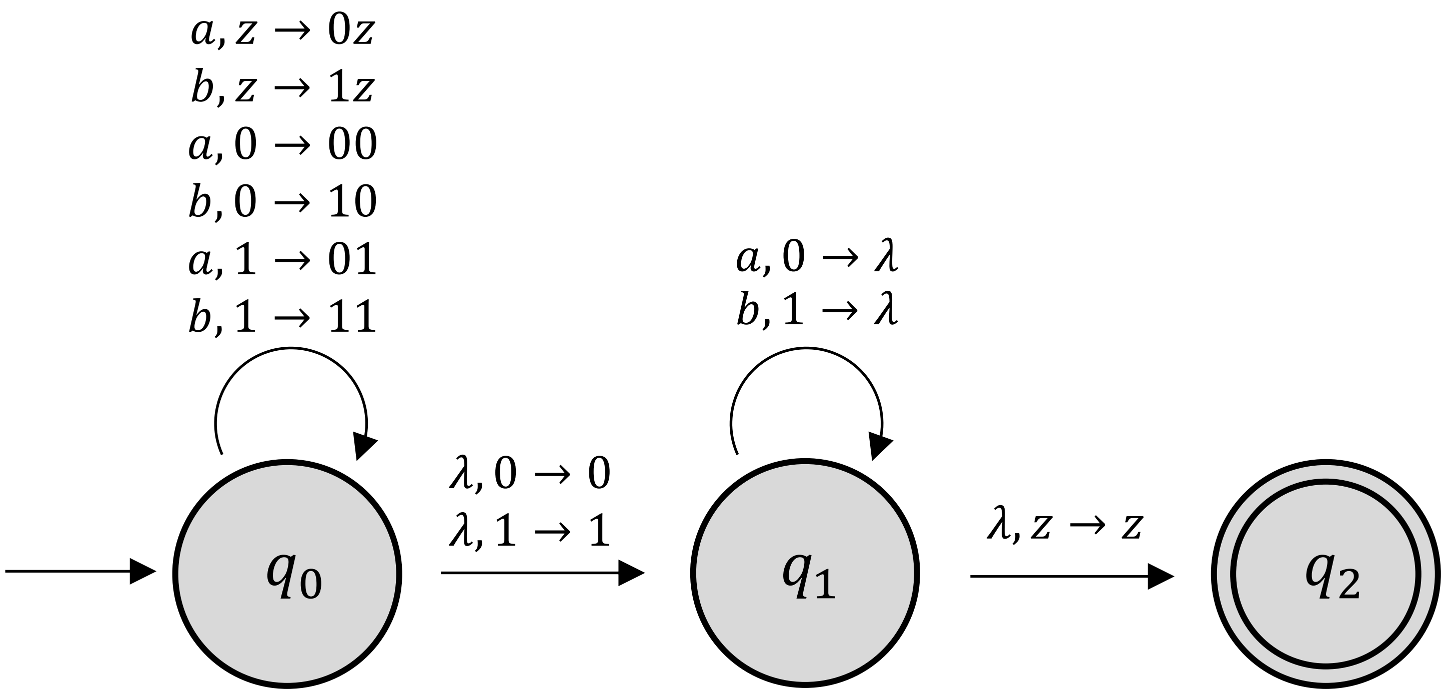

Pushdown Automata

- **Another example:** Design a PDA for $L = \{ww^R : w \in \{a,b\}^*\}$
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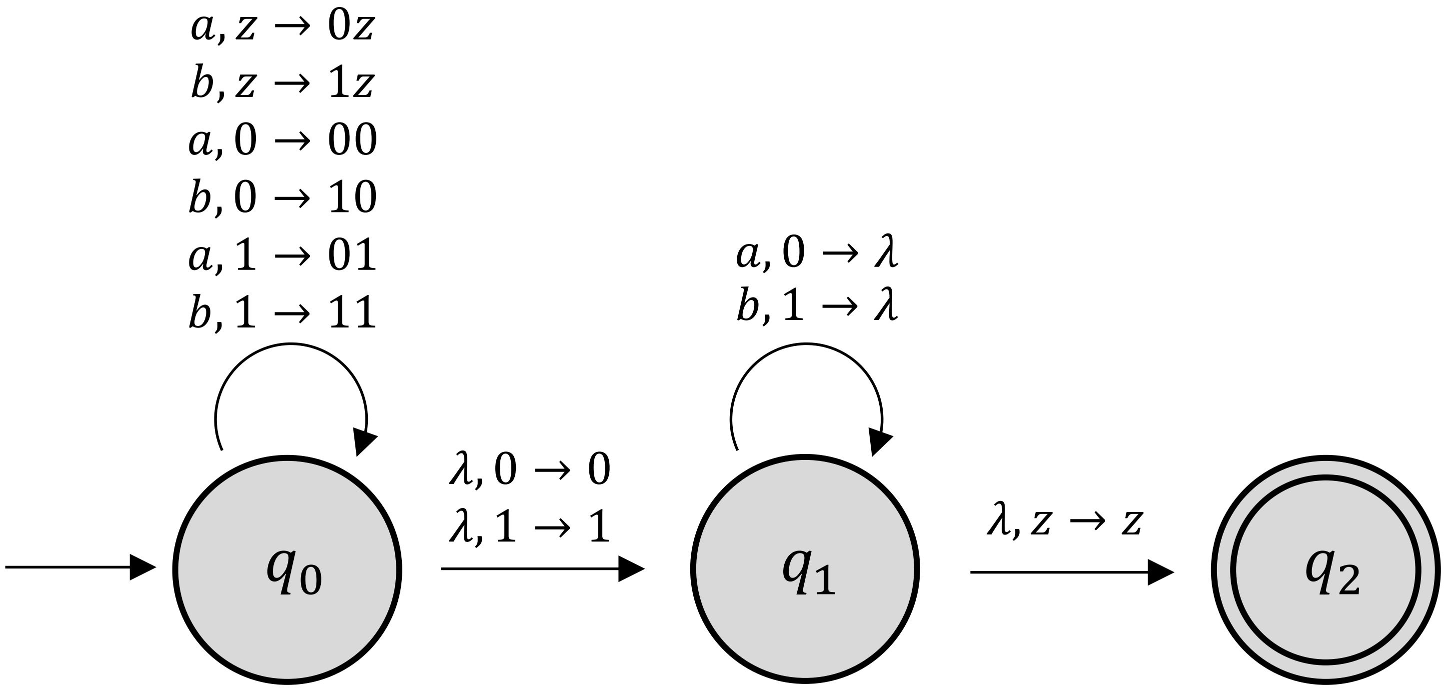

Processing "$abba$"

$$
\begin{aligned}
& (q_0, abba, z) \\
\vdash\ & (q_0, bba, 0z) \\
\vdash\ & (q_0, ba, 10z) \\
\vdash\ & (q_1, ba, 10z) \\
\vdash\ & (q_1, a, 0z) \\
\vdash\ & (q_1, \lambda, z) \\
\vdash\ & (q_2, \lambda, z)
\end{aligned}
$$

Pushdown Automata

- **Another example:** Design a PDA for $L = \{w \in \{a, b\}^* : n_a(w) = n_b(w)\}$

Pushdown Automata

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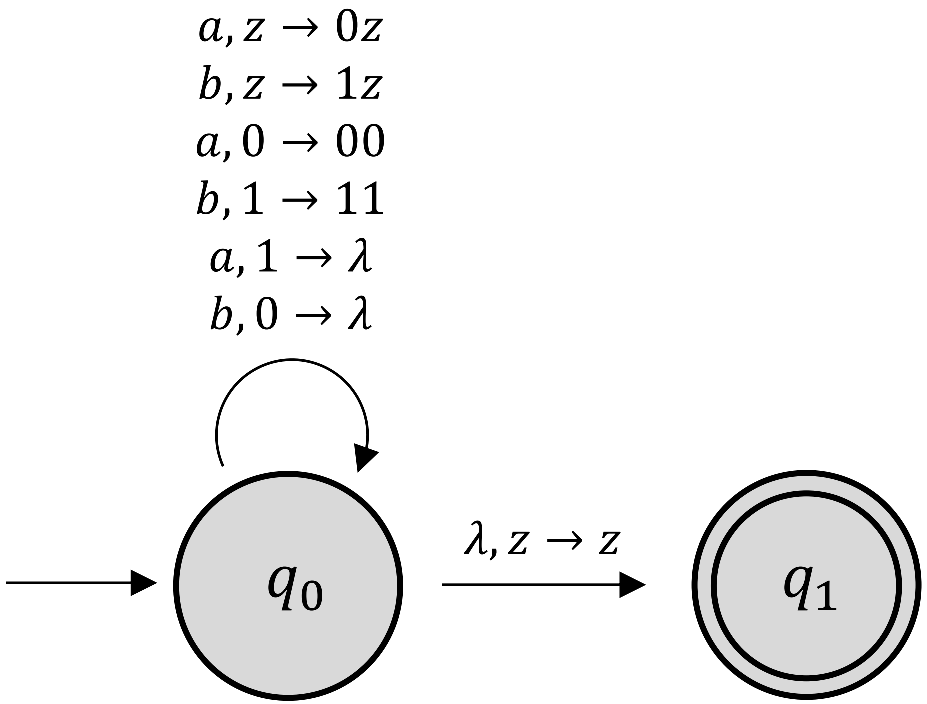
  - $M = (\{q_0, q_1\}, \{a, b\}, \{0,1,z\}, \delta, q_0, z, \{q_1\})$



Pushdown Automata

- **Practice:** Design a PDA for $L = \{a^n b^{2n} : n \geq 0\}$

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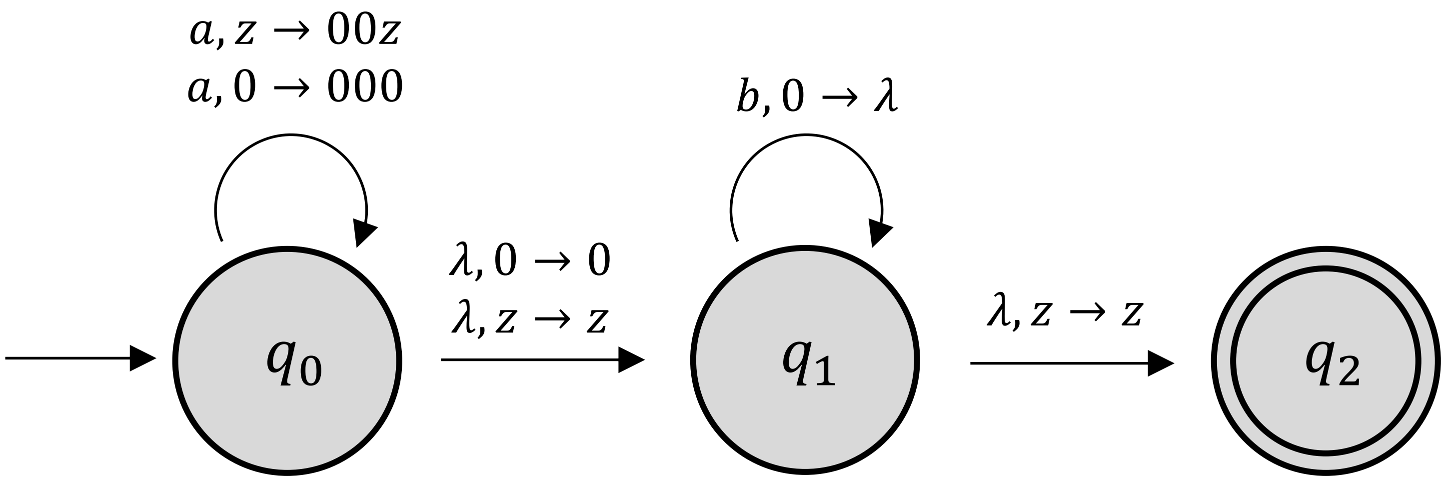

Next Lecture

- **Convert CFG to PDA**
- **Convert PDA to CFG**