Lecture 7

Pushdown Automata

COSE215: Theory of Computation

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Review: Pushdown Automata

- Is it possible to push/pop multiple stack alphabets in one transition?
- Is it possible to push/pop simultaneously in one transition?

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Pushdown Automata

- **(Nondeterministic) Pushdown Automata: Formal definition**
  - A pushdown automaton (PDA) is a 7-tuple: $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$
    - Q is a finite set of **internal states**
    - Σ is a finite set of **symbols**
    - Γ is a finite set of symbols called **stack alphabets**
    - δ is a set of **transition functions**
      - $\delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow 2^{(Q \times \Gamma^*)}$
    - $q_0 \in Q$ is **the initial state**
    - $z \in \Gamma$ is the **initial stack alphabet**
    - $F \subseteq Q$ is a set of **final states**

Review: Pushdown Automata

- Is it possible to push/pop multiple stack alphabets in one transition?
  - YES
- Is it possible to push/pop simultaneously in one transition?
  - YES
- Given a nPDA with multiple push/pop operations, we can generate an equivalent nPDA that pushes/pops one symbol per transition

Contents

- **Convert CFG to PDA**
- **Convert PDA to CFG**

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Assumption
    - CFG is in GNF
    - Consider a leftmost derivation of strings
  - Basic idea
    - Variables in right-hand side → stack
    - Terminals in right-hand side → input

Greibach Normal Forms

- **Definition**
  - Restrict not the length of a production, but the **positions**
  - A CFG (V, T, S, P) is in Greibach normal form if all Ps are of the form
    - $A \rightarrow ax$ ($a \in \Sigma$ and $x \in V^*$)

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Step (1) start symbol → stack

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Step (1) start symbol → stack
  - Step (2) $\forall A \rightarrow ax$
    - Stack: $A \rightarrow x$
    - Input: $a \rightarrow \lambda$

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \to aSbb \mid a$

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \to aSbb \mid a$
    (1) Convert to GNF: $S \to aSA \mid a, A \to bB, B \to b$

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \to aSbb \mid a$

(1) Convert to GNF: $S \to aSA \mid a, A \to bB, B \to b$

(2) $\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$ … start symbol $\to$ stack

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \rightarrow aSbb \mid a$

(1) Convert to GNF: $S \rightarrow aSA \mid a, A \rightarrow bB, B \rightarrow b$

(2) $\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$ … start symbol → stack

(3) $\delta(q_1, a, S) = \{(q_1, SA), (q_1, \lambda)\}$ … $\forall A \rightarrow ax$, Stack: $A \rightarrow x$, Input: $a \rightarrow \lambda$

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \rightarrow aSbb \mid a$

(1) Convert to GNF: $S \rightarrow aSA \mid a, A \rightarrow bB, B \rightarrow b$

(2) $\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$ … start symbol → stack

(3) $\delta(q_1, a, S) = \{(q_1, SA), (q_1, \lambda)\}$ … $\forall A \rightarrow ax$, Stack: $A \rightarrow x$, Input: $a \rightarrow \lambda$

(4) $\delta(q_1, b, A) = \{(q_1, B)\}$ … $\forall A \rightarrow ax$, Stack: $A \rightarrow x$, Input: $a \rightarrow \lambda$

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \to aSbb \mid a$

(1) Convert to GNF: $S \to aSA \mid a, A \to bB, B \to b$

(2) $\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$ … start symbol $\to$ stack

(3) $\delta(q_1, a, S) = \{(q_1, SA), (q_1, \lambda)\}$ … $\forall A \to ax$, Stack: $A \to x$, Input: $a \to \lambda$

(4) $\delta(q_1, b, A) = \{(q_1, B)\}$ … $\forall A \to ax$, Stack: $A \to x$, Input: $a \to \lambda$

(5) $\delta(q_1, b, B) = \{(q_1, \lambda)\}$ … $\forall A \to ax$, Stack: $A \to x$, Input: $a \to \lambda$

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \to aSbb \mid a$

(1) Convert to GNF: $S \to aSA \mid a, A \to bB, B \to b$

(2) $\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$ … start symbol → stack

(3) $\delta(q_1, a, S) = \{(q_1, SA), (q_1, \lambda)\}$ … $\forall A \to ax$, Stack: $A \to x$, Input: $a \to \lambda$

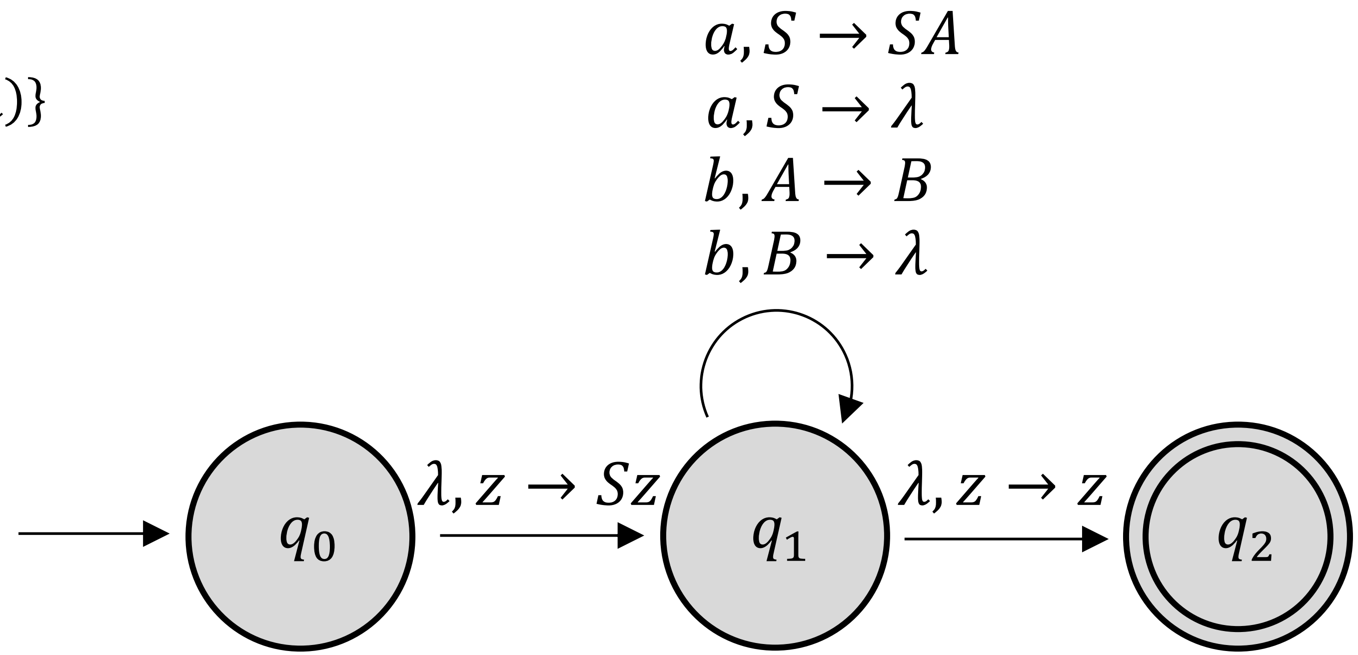
(4) $\delta(q_1, b, A) = \{(q_1, B)\}$ … $\forall A \to ax$, Stack: $A \to x$, Input: $a \to \lambda$

(5) $\delta(q_1, b, B) = \{(q_1, \lambda)\}$ … $\forall A \to ax$, Stack: $A \to x$, Input: $a \to \lambda$

(6) $\delta(q_1, \lambda, z) = \{(q_2, z)\}$ … final state

CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \to aSbb \mid a$
    1. Convert to GNF: $S \to aSA \mid a, A \to bB, B \to b$
    2. $\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$
    3. $\delta(q_1, a, S) = \{(q_1, SA), (q_1, \lambda)\}$
    4. $\delta(q_1, b, A) = \{(q_1, B)\}$
    5. $\delta(q_1, b, B) = \{(q_1, \lambda)\}$
    6. $\delta(q_1, \lambda, z) = \{(q_2, z)\}$



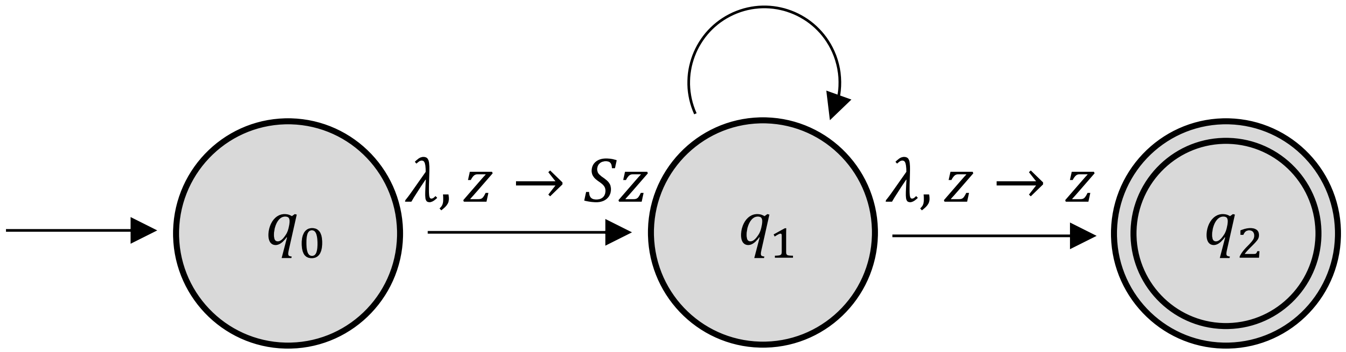
CFG to PDA

- **Generate an nPDA from a given CFG**
  - Example: $S \to aSbb \mid a$
    - Processing "$aaabbbb$"
      - $(q_0, aaabbbb, z)$

$\vdash (q_1, aaabbbb, Sz) \vdash (q_1, aabbbb, SAz)$

$\vdash (q_1, abbbb, SAAz) \vdash (q_1, bbbb, AAz) \vdash (q_1, bbb, BAz)$

$\vdash (q_1, bb, Az) \vdash (q_1, b, Bz) \vdash (q_1, \lambda, z) \vdash (q_2, \lambda, z)$



CFG to PDA

- **Generate an nPDA from a given CFG**
  - How about none-GNF case?

CFG to PDA

- **Generate an nPDA from a given CFG**
  - How about none-GNF case?
  - $G = (V, T, S, P)$
  - $\forall A \rightarrow \alpha$
    - $\delta(q, \lambda, A) \rightarrow \{(q, \alpha)\} \quad (A \in V)$
    - $\delta(q, a, a) \rightarrow \{(q, \lambda)\} \quad (a \in T)$
  - Basic idea
    - Remaining input = stack

CFG to PDA

$\forall A \to \alpha$

- ❖ $\delta(q, \lambda, A) \to \{(q, \alpha)\} \quad (A \in V)$
- ❖ $\delta(q, a, a) \to \{(q, \lambda)\} \quad (a \in T)$

- **Generate an nPDA from a given CFG**
  - Example: $S \to AS \mid \lambda, A \to aAb \mid Ab \mid ab$

CFG to PDA

$\forall A \rightarrow \alpha$

- ❖ $\delta(q, \lambda, A) \rightarrow \{(q, \alpha)\} \quad (A \in V)$
- ❖ $\delta(q, a, a) \rightarrow \{(q, \lambda)\} \quad (a \in T)$

- **Generate an nPDA from a given CFG**
  - Example: $S \rightarrow AS \mid \lambda, A \rightarrow aAb \mid Ab \mid ab$
    - ❖ $\delta(q_0, \lambda, z) \rightarrow \{(q_1, Sz)\}$

CFG to PDA

$\forall A \rightarrow \alpha$

- ❖ $\delta(q, \lambda, A) \rightarrow \{(q, \alpha)\}$ $(A \in V)$
- ❖ $\delta(q, a, a) \rightarrow \{(q, \lambda)\}$ $(a \in T)$

- **Generate an nPDA from a given CFG**
  - Example: $S \rightarrow AS \mid \lambda, A \rightarrow aAb \mid Ab \mid ab$
    - ❖ $\delta(q_0, \lambda, z) \rightarrow \{(q_1, Sz)\}$
    - ❖ $\delta(q_1, \lambda, S) \rightarrow \{(q_1, AS), (q_1, \lambda)\}$
    - ❖ $\delta(q_1, \lambda, A) \rightarrow \{(q_1, aAb), (q_1, Ab), (q_1, ab)\}$

CFG to PDA

$\forall A \to \alpha$

- ❖ $\delta(q, \lambda, A) \to \{(q, \alpha)\}$ $(A \in V)$
- ❖ $\delta(q, a, a) \to \{(q, \lambda)\}$ $(a \in T)$

- **Generate an nPDA from a given CFG**
  - Example: $S \to AS \mid \lambda, A \to aAb \mid Ab \mid ab$
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    - ❖ $\delta(q_1, \lambda, S) \to \{(q_1, AS), (q_1, \lambda)\}$
    - ❖ $\delta(q_1, \lambda, A) \to \{(q_1, aAb), (q_1, Ab), (q_1, ab)\}$
    - ❖ $\delta(q_1, a, a) \to \{(q_1, \lambda)\}$
    - ❖ $\delta(q_1, b, b) \to \{(q_1, \lambda)\}$

CFG to PDA

$\forall A \to \alpha$

- ❖ $\delta(q, \lambda, A) \to \{(q, \alpha)\}$ $(A \in V)$
- ❖ $\delta(q, a, a) \to \{(q, \lambda)\}$ $(a \in T)$

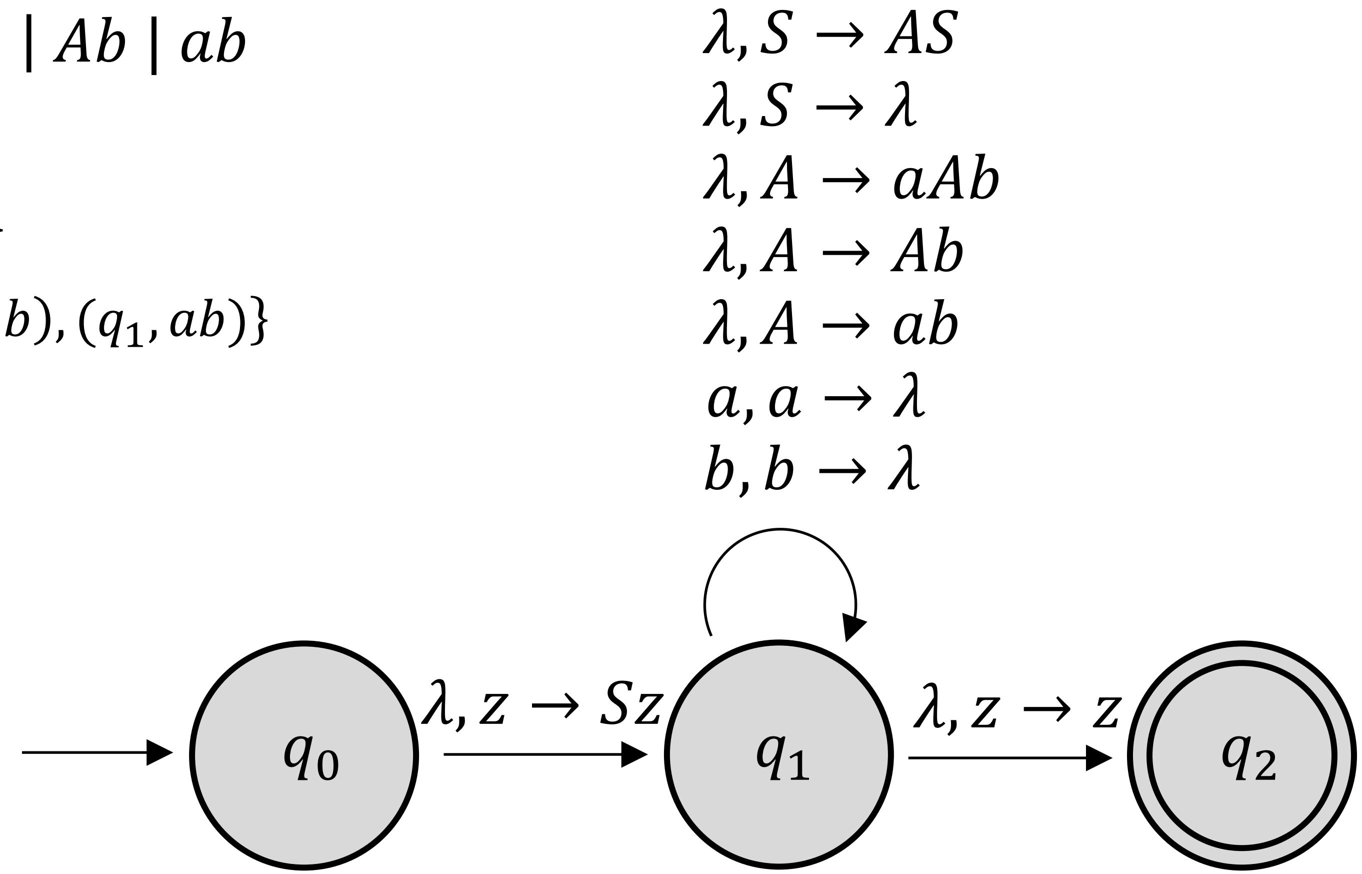
- **Generate an nPDA from a given CFG**
  - Example: $S \to AS \mid \lambda, A \to aAb \mid Ab \mid ab$
    - ❖ $\delta(q_0, \lambda, z) \to \{(q_1, Sz)\}$
    - ❖ $\delta(q_1, \lambda, S) \to \{(q_1, AS), (q_1, \lambda)\}$
    - ❖ $\delta(q_1, \lambda, A) \to \{(q_1, aAb), (q_1, Ab), (q_1, ab)\}$
    - ❖ $\delta(q_1, a, a) \to \{(q_1, \lambda)\}$
    - ❖ $\delta(q_1, b, b) \to \{(q_1, \lambda)\}$
    - ❖ $\delta(q_1, \lambda, z) \to \{(q_2, z)\}$

CFG to PDA

$\forall A \to \alpha$

- ❖ $\delta(q, \lambda, A) \to \{(q, \alpha)\}$ $(A \in V)$
- ❖ $\delta(q, a, a) \to \{(q, \lambda)\}$ $(a \in T)$

- **Generate an nPDA from a given CFG**
  - Example: $S \to AS \mid \lambda, A \to aAb \mid Ab \mid ab$
    - ❖ $\delta(q_0, \lambda, z) \to \{(q_1, Sz)\}$
    - ❖ $\delta(q_1, \lambda, S) \to \{(q_1, AS), (q_1, \lambda)\}$
    - ❖ $\delta(q_1, \lambda, A) \to \{(q_1, aAb), (q_1, Ab), (q_1, ab)\}$
    - ❖ $\delta(q_1, a, a) \to \{(q_1, \lambda)\}$
    - ❖ $\delta(q_1, b, b) \to \{(q_1, \lambda)\}$
    - ❖ $\delta(q_1, \lambda, z) \to \{(q_2, z)\}$



CFG to PDA

$\forall A \to \alpha$

- ❖ $\delta(q, \lambda, A) \to \{(q, \alpha)\}$ $(A \in V)$
- ❖ $\delta(q, a, a) \to \{(q, \lambda)\}$ $(a \in T)$

- **Generate an nPDA from a given CFG**
  - Example: $S \to AS \mid \lambda, A \to aAb \mid Ab \mid ab$
  - Processing "$aabbb$"
    - $(q_0, aabbb, z)$

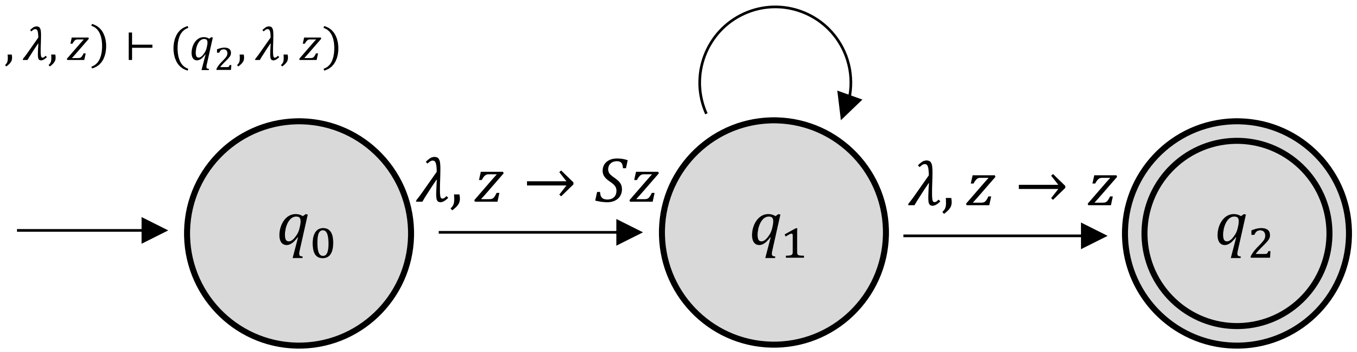
$\vdash (q_1, aabbb, Sz) \vdash (q_1, aabbb, ASz)$

$\vdash (q_1, aabbb, aAbSz) \vdash (q_1, abbb, AbSz) \vdash (q_1, abbb, AbbSz)$

$\vdash (q_1, abbb, abbbSz) \vdash (q_1, bbb, bbbSz) \vdash (q_1, bb, bbSz)$

$\vdash (q_1, b, bSz) \vdash (q_1, \lambda, Sz) \vdash (q_1, \lambda, z) \vdash (q_2, \lambda, z)$

$\lambda, S \to AS$
$\lambda, S \to \lambda$
$\lambda, A \to aAb$
$\lambda, A \to Ab$
$\lambda, A \to ab$
$a, a \to \lambda$
$b, b \to \lambda$



CFG to PDA

$\forall A \to \alpha$

- ❖ $\delta(q, \lambda, A) \to \{(q, \alpha)\}$ $(A \in V)$
- ❖ $\delta(q, a, a) \to \{(q, \lambda)\}$ $(a \in T)$

- **Generate an nPDA from a given CFG: Practice**
  - Example: $S \to \lambda \mid aSb \mid bSa \mid SS$

CFG to PDA

$\forall A \to \alpha$

- ❖ $\delta(q, \lambda, A) \to \{(q, \alpha)\}$ $(A \in V)$
- ❖ $\delta(q, a, a) \to \{(q, \lambda)\}$ $(a \in T)$

- **Generate an nPDA from a given CFG: Practice**
  - Example: $S \to \lambda \mid aSb \mid bSa \mid SS$

PDA to CFG

- **Generate a CFG from a given nPDA**
  - Warning: this is very very complicated

PDA to CFG

- **Generate a CFG from a given nPDA**
  - Warning: this is very very complicated
  - We consider the following type of nPDA
    - An nPDA that has only one final state and can only enter final state when the stack is empty
    - Every transition to a symbol a increases or decreases the contents of the stack by one element
      - E.g., $\delta(q_i, a, A) = (q_j, \lambda)$ or $\delta(q_i, a, A) = (q_j, BC)$
  - Any nPDA can be converted to nPDA that satisfies the above conditions

PDA to CFG

- Step (2) $\forall A \to ax$
  - Stack: $A \to x$
  - Input: $a \to \lambda$

- **Generate a CFG from a given nPDA**
  - Case (1) $\delta(q_i, a, A) = (q_j, \lambda)$
    - CFG has the rule $(q_i A q_j) \to a$
    - $(q_i A q_j)$ is intended to pop A from the stack while going from state q_i to state q_j

PDA to CFG

- Step (2) $\forall A \rightarrow ax$
  - ❖ Stack: $A \rightarrow x$
  - ❖ Input: $a \rightarrow \lambda$

- **Generate a CFG from a given nPDA**
  - Case (2) $\delta(q_i, a, A) = (q_j, BC)$
    - ❖ CFG has the rule $(q_i A q_k) \rightarrow a(q_j B q_l)(q_l C q_k)$
      - q_l and q_k are all possible states belonging to Q
      - $(q_i A q_k) \rightarrow a\big(q_j B q_l\big)(q_l C q_k)$
        - One way to pop A and go from state q_i to state q_k is to read a,
          then use some input to pop B off the stack while going from state q_j to state q_l,
          then read some more input that pops C off the stack and goes from state q_l to state q_k

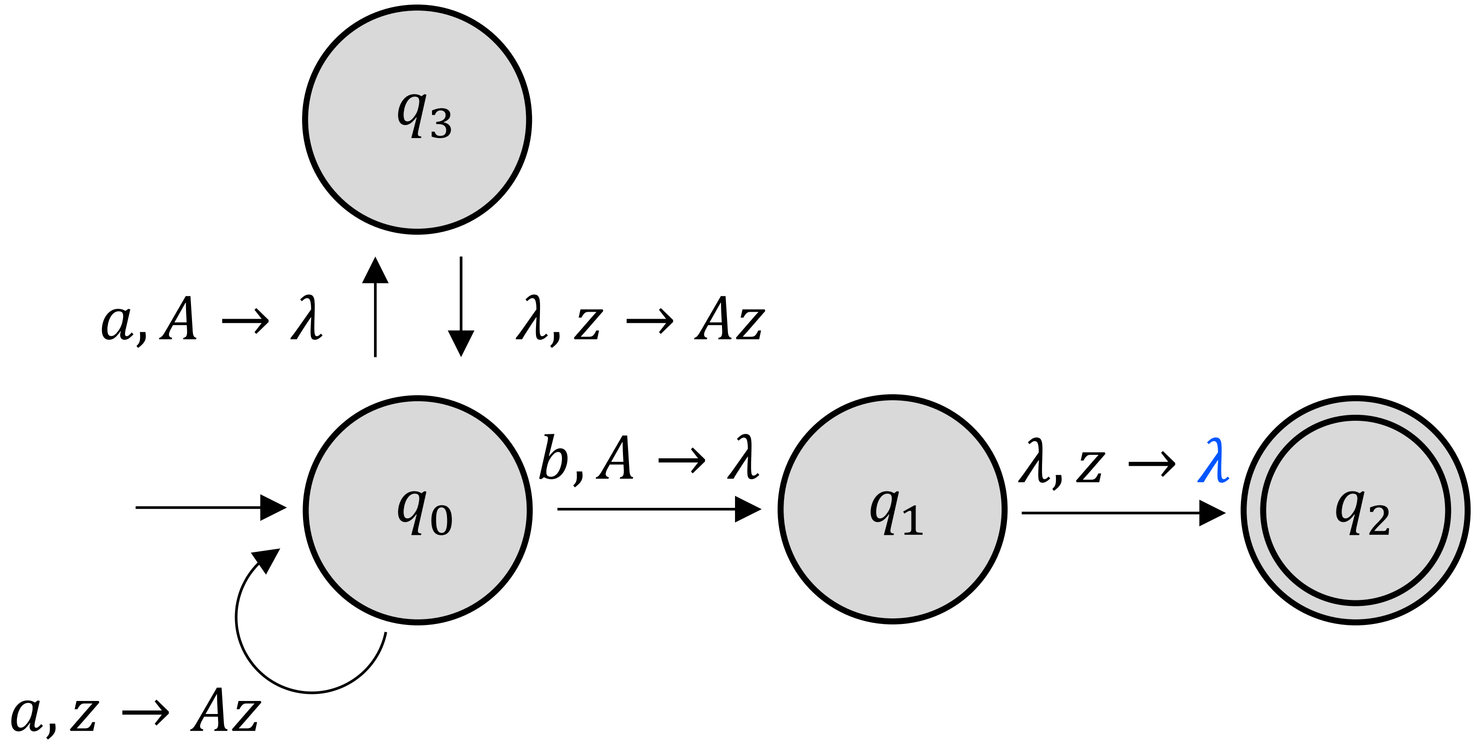
PDA to CFG

- Step (2) $\forall A \to ax$
  - ❖ Stack: $A \to x$
  - ❖ Input: $a \to \lambda$

- **Generate a CFG from a given nPDA**
  - Starting variable = $(q_0 z q_f)$

PDA to CFG

- **Generate a CFG from a given nPDA**
  - Example (q_0: initial state, q_2: final state)
    - ❖ $\delta(q_0, a, z) = \{(q_0, Az)\}$
    - ❖ $\delta(q_3, \lambda, z) = \{(q_0, Az)\}$
    - ❖ $\delta(q_0, a, A) = \{(q_3, \lambda)\}$
    - ❖ $\delta(q_0, b, A) = \{(q_1, \lambda)\}$
    - ❖ $\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$



PDA to CFG

- **Generate a CFG from a given nPDA**
  - Example (q_0: initial state, q_2: final state)
    - $\delta(q_0, a, z) = \{(q_0, Az)\}$
    - $\delta(q_3, \lambda, z) = \{(q_0, Az)\}$
    - $\delta(q_0, a, A) = \{(q_3, \lambda)\}$ => $\boldsymbol{(q_0 A q_3) \rightarrow a}$
    - $\delta(q_0, b, A) = \{(q_1, \lambda)\}$ => $\boldsymbol{(q_0 A q_1) \rightarrow b}$
    - $\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$ => $\boldsymbol{(q_1 z q_2) \rightarrow \lambda}$

- Case (1) $\delta(q_i, a, A) = (q_j, \lambda)$
  - CFG has the rule $(q_i A q_j) \rightarrow a$
- Case (2) $\delta(q_i, a, A) = (q_j, BC)$
  - CFG has the rule $(q_i A q_k) \rightarrow a(q_j B q_l)(q_l C q_k)$
    - q_l and q_k are all possible states belonging to Q

PDA to CFG

- Case (2) $\delta(q_i, a, A) = (q_j, BC)$
  - ❖ CFG has the rule $(q_i A q_k) \rightarrow a(q_j B q_l)(q_l C q_k)$
    - q_l and q_k are all possible states belonging to Q

- **Generate a CFG from a given nPDA**
  - Example (q_0: initial state, q_2: final state)

    - ❖ $\delta(q_0, a, z) = \{(q_0, Az)\}$
    - ❖ $(q_0 z q_0) \rightarrow a(q_0 A q_0)(q_0 z q_0) \mid a(q_0 A q_1)(q_1 z q_0) \mid a(q_0 A q_2)(q_2 z q_0) \mid a(q_0 A q_3)(q_3 z q_0),$
    - ❖ $(q_0 z q_1) \rightarrow a(q_0 A q_0)(q_0 z q_1) \mid a(q_0 A q_1)(q_1 z q_1) \mid a(q_0 A q_2)(q_2 z q_1) \mid a(q_0 A q_3)(q_3 z q_1),$
    - ❖ $(q_0 z q_2) \rightarrow a(q_0 A q_0)(q_0 z q_2) \mid a(q_0 A q_1)(q_1 z q_2) \mid a(q_0 A q_2)(q_2 z q_2) \mid a(q_0 A q_3)(q_3 z q_2),$
    - ❖ $(q_0 z q_3) \rightarrow a(q_0 A q_0)(q_0 z q_3) \mid a(q_0 A q_1)(q_1 z q_3) \mid a(q_0 A q_2)(q_2 z q_3) \mid a(q_0 A q_3)(q_3 z q_3)$

PDA to CFG

- Case (2) $\delta(q_i, a, A) = (q_j, BC)$
  - ❖ CFG has the rule $(q_i A q_k) \to a(q_j B q_l)(q_l C q_k)$
    - q_l and q_k are all possible states belonging to Q

- **Generate a CFG from a given nPDA**

  - Example (q_0: initial state, q_2: final state)
    - ❖ $\delta(q_3, \lambda, z) = \{(q_0, Az)\}$
    - ❖ $(q_3 z q_0) \to (q_0 A q_0)(q_0 z q_0) \mid (q_0 A q_1)(q_1 z q_0) \mid (q_0 A q_2)(q_2 z q_0) \mid (q_0 A q_3)(q_3 z q_0),$
    - ❖ $(q_3 z q_1) \to (q_0 A q_0)(q_0 z q_1) \mid (q_0 A q_1)(q_1 z q_1) \mid (q_0 A q_2)(q_2 z q_1) \mid (q_0 A q_3)(q_3 z q_1),$
    - ❖ $(q_3 z q_2) \to (q_0 A q_0)(q_0 z q_2) \mid (q_0 A q_1)(q_1 z q_2) \mid (q_0 A q_2)(q_2 z q_2) \mid (q_0 A q_3)(q_3 z q_2),$
    - ❖ $(q_3 z q_3) \to (q_0 A q_0)(q_0 z q_3) \mid (q_0 A q_1)(q_1 z q_3) \mid (q_0 A q_2)(q_2 z q_3) \mid (q_0 A q_3)(q_3 z q_3)$

PDA to CFG

- **Generate a CFG from a given nPDA**
  - Example (q_0: initial state, q_2: final state)
    - ❖ $(q_0Aq_3) \to a,$
    - ❖ $(q_0Aq_1) \to b,$
    - ❖ $(q_1zq_2) \to \lambda,$
    - ❖ $(q_0zq_0) \to a(q_0Aq_0)(q_0zq_0) \mid a(q_0Aq_1)(q_1zq_0) \mid a(q_0Aq_2)(q_2zq_0) \mid a(q_0Aq_3)(q_3zq_0),$
    - ❖ $(q_0zq_1) \to a(q_0Aq_0)(q_0zq_1) \mid a(q_0Aq_1)(q_1zq_1) \mid a(q_0Aq_2)(q_2zq_1) \mid a(q_0Aq_3)(q_3zq_1),$
    - ❖ $(q_0zq_2) \to a(q_0Aq_0)(q_0zq_2) \mid a(q_0Aq_1)(q_1zq_2) \mid a(q_0Aq_2)(q_2zq_2) \mid a(q_0Aq_3)(q_3zq_2),$
    - ❖ $(q_0zq_3) \to a(q_0Aq_0)(q_0zq_3) \mid a(q_0Aq_1)(q_1zq_3) \mid a(q_0Aq_2)(q_2zq_3) \mid a(q_0Aq_3)(q_3zq_3),$
    - ❖ $(q_3zq_0) \to (q_0Aq_0)(q_0zq_0) \mid (q_0Aq_1)(q_1zq_0) \mid (q_0Aq_2)(q_2zq_0) \mid (q_0Aq_3)(q_3zq_0),$
    - ❖ $(q_3zq_1) \to (q_0Aq_0)(q_0zq_1) \mid (q_0Aq_1)(q_1zq_1) \mid (q_0Aq_2)(q_2zq_1) \mid (q_0Aq_3)(q_3zq_1),$
    - ❖ $(q_3zq_2) \to (q_0Aq_0)(q_0zq_2) \mid (q_0Aq_1)(q_1zq_2) \mid (q_0Aq_2)(q_2zq_2) \mid (q_0Aq_3)(q_3zq_2),$
    - ❖ $(q_3zq_3) \to (q_0Aq_0)(q_0zq_3) \mid (q_0Aq_1)(q_1zq_3) \mid (q_0Aq_2)(q_2zq_3) \mid (q_0Aq_3)(q_3zq_3)$

PDA to CFG

unnecessary variable
$(q_0 A q_0)$
$(q_0 A q_2)$

No transitions
1->0
1->1
1->3
2->2

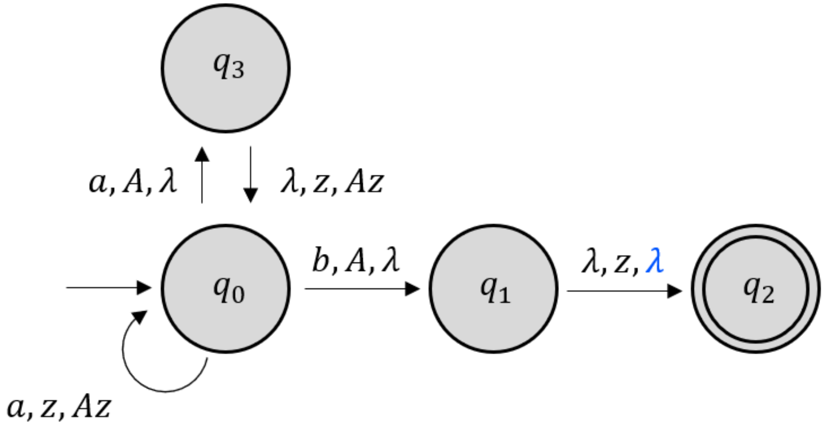
- **Generate a CFG from a given nPDA**
  - Example (q_0: initial state, q_2: final state)
    - ❖ $(q_0 A q_3) \to a,$
    - ❖ $(q_0 A q_1) \to b,$
    - ❖ $(q_1 z q_2) \to \lambda,$
    - ❖ $(q_0 z q_0) \to \cancel{a(q_0 A q_0)(q_0 z q_0) \mid a(q_0 A q_1)(q_1 z q_0) \mid a(q_0 A q_2)(q_2 z q_0)} \mid a(q_0 A q_3)(q_3 z q_0),$
    - ❖ $(q_0 z q_1) \to \cancel{a(q_0 A q_0)(q_0 z q_1) \mid a(q_0 A q_1)(q_1 z q_1) \mid a(q_0 A q_2)(q_2 z q_1)} \mid a(q_0 A q_3)(q_3 z q_1),$
    - ❖ $(q_0 z q_2) \to \cancel{a(q_0 A q_0)(q_0 z q_2)} \mid a(q_0 A q_1)(q_1 z q_2) \mid \cancel{a(q_0 A q_2)(q_2 z q_2)} \mid a(q_0 A q_3)(q_3 z q_2),$
    - ❖ $(q_0 z q_3) \to \cancel{a(q_0 A q_0)(q_0 z q_3) \mid a(q_0 A q_1)(q_1 z q_3) \mid a(q_0 A q_2)(q_2 z q_3)} \mid a(q_0 A q_3)(q_3 z q_3),$
    - ❖ $(q_3 z q_0) \to \cancel{(q_0 A q_0)(q_0 z q_0) \mid (q_0 A q_1)(q_1 z q_0) \mid (q_0 A q_2)(q_2 z q_0)} \mid (q_0 A q_3)(q_3 z q_0),$
    - ❖ $(q_3 z q_1) \to \cancel{(q_0 A q_0)(q_0 z q_1) \mid (q_0 A q_1)(q_1 z q_1) \mid (q_0 A q_2)(q_2 z q_1)} \mid (q_0 A q_3)(q_3 z q_1),$
    - ❖ $(q_3 z q_2) \to \cancel{(q_0 A q_0)(q_0 z q_2)} \mid (q_0 A q_1)(q_1 z q_2) \mid \cancel{(q_0 A q_2)(q_2 z q_2)} \mid (q_0 A q_3)(q_3 z q_2),$
    - ❖ $(q_3 z q_3) \to \cancel{(q_0 A q_0)(q_0 z q_3) \mid (q_0 A q_1)(q_1 z q_3) \mid (q_0 A q_2)(q_2 z q_3)} \mid (q_0 A q_3)(q_3 z q_3)$

PDA to CFG

- **Generate a CFG from a given nPDA**
  - Example (q_0: initial state, q_2: final state)
    - $(q_0 A q_3) \to a,$
    - $(q_0 A q_1) \to b,$
    - $(q_1 z q_2) \to \lambda,$
    - $(q_0 z q_0) \to a(q_0 A q_3)(q_3 z q_0),$
    - $(q_0 z q_1) \to a(q_0 A q_3)(q_3 z q_1),$
    - $(q_0 z q_2) \to a(q_0 A q_1)(q_1 z q_2) \mid a(q_0 A q_3)(q_3 z q_2),$
    - $(q_0 z q_3) \to a(q_0 A q_3)(q_3 z q_3),$
    - $(q_3 z q_0) \to (q_0 A q_3)(q_3 z q_0),$
    - $(q_3 z q_1) \to (q_0 A q_3)(q_3 z q_1),$
    - $(q_3 z q_2) \to (q_0 A q_1)(q_1 z q_2) \mid (q_0 A q_3)(q_3 z q_2),$
    - $(q_3 z q_3) \to (q_0 A q_3)(q_3 z q_3)$

PDA to CFG

- **Generate a CFG from a given nPDA**
  - Example (q_0: initial state, q_2: final state)
    - ❖ $(q_0Aq_3) \to a,$
    - ❖ $(q_0Aq_1) \to b,$
    - ❖ $(q_1zq_2) \to \lambda,$
    - ❖ $(q_0zq_0) \to a(q_0Aq_3)(q_3zq_0),$
    - ❖ $(q_0zq_1) \to a(q_0Aq_3)(q_3zq_1),$
    - ❖ $(q_0zq_2) \to a(q_0Aq_1)(q_1zq_2) \mid a(q_0Aq_3)(q_3zq_2),$
    - ❖ $(q_0zq_3) \to a(q_0Aq_3)(q_3zq_3),$
    - ❖ $(q_3zq_0) \to (q_0Aq_3)(q_3zq_0),$
    - ❖ $(q_3zq_1) \to (q_0Aq_3)(q_3zq_1),$
    - ❖ $(q_3zq_2) \to (q_0Aq_1)(q_1zq_2) \mid (q_0Aq_3)(q_3zq_2),$
    - ❖ $(q_3zq_3) \to (q_0Aq_3)(q_3zq_3)$



Deriving "aab“

$$
\begin{aligned}
(q_0zq_2) &\Rightarrow a(q_0Aq_3)(q_3zq_2) \\
&\Rightarrow aa(q_3zq_2) \\
&\Rightarrow aa(q_0Aq_1)(q_1zq_2) \\
&\Rightarrow aab(q_1zq_2) \\
&\Rightarrow aab
\end{aligned}
$$

Context-Free Languages

- **If $L = L(M)$ for some NPDA M, then L is a context-free language**
- **A language is context-free iff it is accepted by some NPDA**
  - For every CFG, there exists an equivalent NPDA
  - For every NPDA, there exists an equivalent CFG

Next Lecture

- **Deterministic Pushdown Automata**