Lecture 8

Properties of Context-free Languages

COSE215: Theory of Computation

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Pumping lemma for context-free languages

- **Pumping lemma for CFL: example4**
  - $L = \{a^{n!}: n \geq 0\}$
    - ❖The adversary picks m, and suppose we pick $w = a^{m!}$
    - ❖The adversary breaks $w = uvxyz$, where $|vxy| \leq m$ and $|vy| \geq 1$

Pumping lemma for context-free languages

- **Pumping lemma for CFL: practice**
  - $L = \{a^n b a^n b a^n \mid n \geq 0\}$

Pumping lemma for context-free languages

- **Pumping lemma for CFL: practice**
  - $L = \{a^n b^j : n = j^2\}$

Contents

- **Closure properties of context-free languages**

Closure properties of CFLs

- **Revisit: closure properties of regular languages**
  - If L_1 and L_2 are regular languages, then so are
    - $L_1 \cup L_2$ (UNION)
    - $L_1 \cap L_2$ (INTERSECTION)
    - $L_1 \cdot L_2$ (CONCATENATION)
    - $L_1 - L_2$ (DIFFERENCE)
    - $\overline{L_1}$ (COMPLEMENTATION)
    - L_1^* (STAR)

Closure properties of CFLs

- **Revisit: closure properties of regular languages**
  - If L_1 and L_2 are regular languages, then so are
    - $L_1 \cup L_2$ (UNION)
    - $L_1 \cap L_2$ (INTERSECTION)
    - $L_1 \cdot L_2$ (CONCATENATION)
    - $L_1 - L_2$ (DIFFERENCE)
    - $\overline{L_1}$ (COMPLEMENTATION)
    - L_1^* (STAR)
- **What about in the CFL?**

Closure properties of CFLs

- **Case 1) UNION**
  - If L_1 and L_2 are CFLs, is $L_1 \cup L_2$ also a CFL?

Closure properties of CFLs

- **Case 1) UNION**
  - If L_1 and L_2 are CFLs, is $L_1 \cup L_2$ also a CFL?
  - Let $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$ be context-free grammars (with $V_1 \cap V_2 = \emptyset$)

Closure properties of CFLs

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  - If L_1 and L_2 are CFLs, is $L_1 \cup L_2$ also a CFL?
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  - Consider the following CFG
    - $G_3 = (V_1 \cup V_2 \cup \{S_3\}, T_1 \cup T_2, S_3, P_3)$, where $P_3 = P_1 \cup P_2 \cup \{S_3 \rightarrow S_1 | S_2\}$

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    - ❖ Every sentence generated by L_1 ($w_1 \in L_1$) can also be generated by $S_3 \Rightarrow S_1 \overset{*}{\Rightarrow} w_1$
    - ❖ Every sentence generated by L_2 ($w_2 \in L_2$) can also be generated by $S_3 \Rightarrow S_2 \overset{*}{\Rightarrow} w_2$

Closure properties of CFLs

- **Case 1) UNION**
  - If L_1 and L_2 are CFLs, is $L_1 \cup L_2$ also a CFL?
  - Let $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$ be context-free grammars (with $V_1 \cap V_2 = \emptyset$)
  - Consider the following CFG
    - ❖ $G_3 = (V_1 \cup V_2 \cup \{S_3\}, T_1 \cup T_2, S_3, P_3)$, where $P_3 = P_1 \cup P_2 \cup \{S_3 \rightarrow S_1 | S_2\}$
    - ❖ Then the CFL $L(G_3) = L_1 \cup L_2$
    - ❖ CFL is closed under union

Closure properties of CFLs

- **Case 2) CONCATENATION**
  - If L_1 and L_2 are CFLs, is $L_1 L_2$ also a CFL?

Closure properties of CFLs

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    - ❖ Any $w_1 w_2$ ($w_1 \in L_1$ and $w_2 \in L_2$) can be generated by $S_3 \Rightarrow S_1 S_2 \stackrel{*}{\Rightarrow} w_1 w_2$

Closure properties of CFLs

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  - If L_1 and L_2 are CFLs, is $L_1 L_2$ also a CFL?
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  - Consider the following CFG
    - ❖ $G_3 = (V_1 \cup V_2 \cup \{S_3\}, T_1 \cup T_2, S_3, P_3)$, where $P_3 = P_1 \cup P_2 \cup \{S_3 \to S_1 S_2\}$
    - ❖ Any $w_1 w_2$ ($w_1 \in L_1$ and $w_2 \in L_2$) can be generated by $S_3 \Rightarrow S_1 S_2 \overset{*}{\Rightarrow} w_1 w_2$
    - ❖ Then the CFL $L(G_3) = L_1 L_2$
    - ❖ CFL is closed under concatenation

Closure properties of CFLs

- **Case 3) STAR**
  - If L is a CFL, is L^* also a CFL?

Closure properties of CFLs

- **Case 3) STAR**
  - If L is a CFL, is L^* also a CFL?
  - Let $G = (V, T, S, P)$ be a context-free grammar

Closure properties of CFLs

- **Case 3) STAR**
  - If L is a CFL, is L^* also a CFL?
  - Let $G = (V, T, S, P)$ be a context-free grammar
  - Consider the following CFG
    - ❖ $G' = (V \cup \{S'\}, T, S', P')$, where $P' = P \cup \{S' \rightarrow SS' \mid \lambda\}$

Closure properties of CFLs

- **Case 3) STAR**
  - If L is a CFL, is L^* also a CFL?
  - Let $G = (V, T, S, P)$ be a context-free grammar
  - Consider the following CFG
    - ❖ $G' = (V \cup \{S'\}, T, S', P')$, where $P' = P \cup \{S' \rightarrow SS' \mid \lambda\}$
    - ❖ Then the CFL $L(G') = L(G)^*$
    - ❖ CFL is closed under star

Closure properties of CFLs

- **Case 4) INTERSECTION**
  - If L_1 and L_2 are CFLs, is $L_1 \cap L_2$ also a CFL?

Closure properties of CFLs

- **Case 4) INTERSECTION**
  - If L_1 and L_2 are CFLs, is $L_1 \cap L_2$ also a CFL?
  - No!
  - Counter example: consider the following two CFLs
    - $L_1 = \{a^n b^n c^m : n \geq 0, m \geq 0\}$
    - $L_2 = \{a^m b^n c^n : n \geq 0, m \geq 0\}$

Closure properties of CFLs

- **Case 4) INTERSECTION**
  - If L_1 and L_2 are CFLs, is $L_1 \cap L_2$ also a CFL?
  - No!
  - Counter example: consider the following two CFLs
    - ❖ $L_1 = \{a^n b^n c^m : n \geq 0, m \geq 0\}$
    - ❖ $L_2 = \{a^m b^n c^n : n \geq 0, m \geq 0\}$
    - ❖ e.g., Production rules for L_1
      - $S \rightarrow S_1 S_2$
      - $S_1 \rightarrow a S_1 b \mid \lambda$
      - $S_2 \rightarrow c S_2 \mid \lambda$

Closure properties of CFLs

- **Case 4) INTERSECTION**
  - If L_1 and L_2 are CFLs, is $L_1 \cap L_2$ also a CFL?
  - No!
  - Counter example: consider the following two CFLs
    - $L_1 = \{a^n b^n c^m : n \geq 0, m \geq 0\}$
    - $L_2 = \{a^m b^n c^n : n \geq 0, m \geq 0\}$
    - $L_1 \cap L_2 = \{a^n b^n c^n : n \geq 0\}$, which is not a CFL
    - CFL is not closed under intersection

Closure properties of CFLs

- **Case 5) COMPLEMENTATION**
  - If L is a CFL, is $\overline{L}$ also a CFL?

Closure properties of CFLs

- **Case 5) COMPLEMENTATION**
  - If L is a CFL, is $\overline{L}$ also a CFL?
  - No!
  - Suppose: CFL is closed under complementation
    - If L_1 and L_2 are CFLs, then $\overline{L_1} \cup \overline{L_2}$ should be a CFL

Closure properties of CFLs

- **Case 5) COMPLEMENTATION**
  - If L is a CFL, is $\overline{L}$ also a CFL?
  - No!
  - Suppose: CFL is closed under complementation
    - If L_1 and L_2 are CFLs, then $\overline{L_1} \cup \overline{L_2}$ should be a CFL
    - $\overline{\overline{L_1} \cup \overline{L_2}}$ should be a CFL

Closure properties of CFLs

- **Case 5) COMPLEMENTATION**
  - If L is a CFL, is $\overline{L}$ also a CFL?
  - No!
  - Suppose: CFL is closed under complementation
    - If L_1 and L_2 are CFLs, then $\overline{L_1} \cup \overline{L_2}$ should be a CFL
    - $\overline{\overline{L_1} \cup \overline{L_2}}$ should be a CFL
    - $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$ // CONTRADICTION
    - CFL is not closed under complementation

Closure properties of CFLs

- **Case 6) DIFFERENCE**
  - If L_1 and L_2 are CFLs, is $L_1 - L_2$ also a CFL?

Closure properties of CFLs

- **Case 6) DIFFERENCE**
  - If L_1 and L_2 are CFLs, is $L_1 - L_2$ also a CFL?
  - No!
  - Suppose: CFL is closed under difference

Closure properties of CFLs

- **Case 6) DIFFERENCE**
  - If L_1 and L_2 are CFLs, is $L_1 - L_2$ also a CFL?
  - No!
  - Suppose: CFL is closed under difference
    - $L_1 - L_2$ should be a CFL
    - $L_1 - (L_1 - L_2)$ should be a CFL

Closure properties of CFLs

- **Case 6) DIFFERENCE**
  - If L_1 and L_2 are CFLs, is $L_1 - L_2$ also a CFL?
  - No!
  - Suppose: CFL is closed under difference
    - ❖ $L_1 - L_2$ should be a CFL
    - ❖ $L_1 - (L_1 - L_2)$ should be a CFL
    - ❖ $L_1 - (L_1 - L_2) = L_1 \cap L_2$ // CONTRADICTION
    - ❖ CFL is not closed under difference

Closure properties of CFLs

	RL	CFL
Union	O	O
Concatenation	O	O
Star	O	O
Intersection	O	X
Complementation	O	X
Difference	O	X

Next Lecture

- **Turing machine**